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On the problem of convexity for the restricted three-body problem around the heavy primary

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Abstract. In this note, we prove that below the first critical energy level, a proper combination of the Ligon-Schaaf and Levi-Civita regularization mappings provides a convex symplectic embedding of the double cover of the energy surfaces of the planar rotating Kepler problem into $\mathbb{R}^4$ endowed with its standard symplectic structure. This convex embedding extends to the bounded component of the planar circular restricted three-body problem around the heavy body outside a small neighborhood of the collisions. This opens up new approaches to attack the Birkhoff conjecture about the existence of a global surface of section in the restricted planar circular three-body problem using holomorphic curve techniques.

Key words: Global surfaces of section, convexity, restricted three-body problem, contact and symplectic topology.

1. Introduction

The restricted three-body problem describes the motion of a massless particle in the presence of two massive primaries whose masses we denote by $1-\mu$ and μ, and which move along an orbit in the two-body problem.

In this article, we will consider the circular, planar restricted three-body problem (RTBP), which means that we assume that the primaries move in circular orbits around each other and the particle is coplanar with them. Remarkably, with these assumptions, one can describe this problem in proper uniformly-rotating coordinates with an *autonomous* Hamiltonian. This Hamiltonian, sometimes referred to as Jacobi integral, is the function $H : T^*\mathbb{R}^2 \setminus \{(-\mu, 0), (1-\mu, 0)\} \to \mathbb{R}$ given by

$$H(q,p) = \frac{1}{2}\|p\|^2 + (q_1p_2 - q_2p_1) - \frac{1-\mu}{\|q+(\mu,0)\|} - \frac{\mu}{\|q-(1-\mu,0)\|}. \qquad (1)$$

See [14, Chapter 5.1–5.3] for a derivation of this Hamiltonian. For a mass ratio $\mu \in (0,1)$, the Jacobi Hamiltonian has five critical points, commonly referred to as Lagrange points, see [14, Chapter 5.4] or [1, Chapter 10.2]. As explained in these references, three of these critical points have Morse index equal to 1, and two of them have index equal to 2. Some references, such as [16, Definition 7.6], refer to the critical points of index 1 as Euler points (these are limits of three-body Euler central configurations when one body becomes massless); we will follow the more conventional labeling as Lagrange points. A detailed explanation of the relation of the Lagrange points with the topology of the level sets can be found in [14, Chapter 5.4].

In this paper we will be concerned with level sets

$$\{H = c\}$$

where c is smaller than the first critical value $L_1(\mu)$. In this case, the level set

$$\Sigma_{\mu,c} := H^{-1}(c)$$

has three components for $\mu \in (0, 1)$, all of which are non-compact and non-convex. This is explained in Appendix A.

Our goal is to describe the dynamics on one of the components, namely the one around the heavy primary by a *convex* Hamiltonian on $\mathbb{R}^4$ and in this paper we achieve this outside a small neighborhood of the collisions. Note that such a description by a convex Hamiltonian is of great help in understanding the dynamics, since convex Hamiltonians enjoy many good properties. For instance, one can apply Morse theory as for example in Ekeland's book [11]. More recently, Hofer-Wysocki-Zehnder showed, using holomorphic curve techniques, that convex Hamiltonians admit disk-like global surfaces of section [20]. We will review some recent results in this direction in Section 4.

To find a description with a convex Hamiltonian, one needs a suitable regularization scheme. For the Kepler problem, some of the best known schemes are Levi-Civita, Ligon-Schaaf and Moser regularizations. The first scheme directly provides a Hamiltonian defined on $\mathbb{R}^4$ with level sets having a component diffeomorphic to S^3. Unfortunately, this component turns out to be non-convex in general as was shown in [3, Theorem 1.2]. The other two schemes also regularize collision orbits, but they compactify one component of the level set to the unit cotangent bundle of S^2 rather than S^3. Hence by doing so we cannot obtain a convex set in $\mathbb{R}^4$ since the unit cotangent bundle of S^2 does not even embed in $\mathbb{R}^4$ as a topological manifold. Appendix D of [8] contains three different proofs of this fact. For an older reference, see Satz II from [21].

We will hence consider, similar to Levi-Civita regularization, the double cover $p : S^3 \to ST^*S^2$, the unit cotangent bundle of the 2-sphere, but we will use a different embedding $i : S^3 \to \mathbb{R}^4$ and a different Hamiltonian $\tilde{H}_r$ than the one coming from Levi-Civita regularization. The embedding that we construct has the following properties for $\mu = 0$.

- the image of the embedding $i : S^3 \to \mathbb{R}^4$ bounds a strictly convex set.
- there is a Hamiltonian $\tilde{H}_r : \mathbb{R}^4 \to \mathbb{R}$ such that $i(S^3)$ is a component of $\tilde{H}_r^{-1}(0)$
- furthermore, the Hamiltonian vector field $X_{\tilde{H}_r}$ projects under p to the Hamiltonian vector field of the Ligon-Schaaf regularization.

Our strategy is the following. We first consider the limit case $\mu = 0$ of the restricted three-body problem, which is just the rotating Kepler problem. We show that the Ligon-Schaaf regularization scheme can be combined with the Levi-Civita map and find that, for all energies below the first critical value, a component of the level set of the rotating Kepler problem is strictly convex. This is done by explicitly computing the Gauss-Kronecker curvature, which turns out to be a polynomial that factorizes surprisingly well. This allows us to show that the curvature is positive. Outside a neighborhood of the collisions, the perturbing function is smooth, which allows us to extend the convexity result to this case as well. Due to the singularity of the perturbing function at the collisions we haven't been able to show that such a convexity result holds in a neighborhood of the collisions for small $\mu > 0$.

To state the theorem, we introduce the following notation. Write $\pi : (q,p) \mapsto q$ for the footpoint projection, and define $\Sigma^b_{\mu,c}$ as the component of the level set $\Sigma_{\mu,c}$ for which the closure of the projection $\pi(\Sigma^b_{\mu,c})$ contains the point $(-\mu, 0)$. In more intuitive language, $\Sigma^b_{\mu,c}$ is the bounded component containing the primary with mass $1-\mu$. We write the 2-fold cover of $\Sigma^b_{\mu,c}$ as $\tilde{\Sigma}^b_{\mu,c}$. We then have

Theorem A *For all c below the first critical value $L_1(0) = -3/2$ of the rotating Kepler problem, there is an embedding map $i : \tilde{\Sigma}^b_{0,c} \to \mathbb{R}^4$, and a Hamiltonian $\tilde{H}_r$ with the following properties*:

(1) *the image* $i(\tilde{\Sigma}^b_{0,c})$ *is contained in a component of* $\tilde{H}_r^{-1}(c)$.
(2) *the closure of the image* $i(\tilde{\Sigma}^b_{0,c})$ *bounds a smooth, strictly convex set.*
(3) *the Hamiltonian vector field* $X_{\tilde{H}_r}$ *is tangent to* $i(\tilde{\Sigma}^b_{0,c})$ *and is a reparametrization of* X_H *in the sense that the local diffeomorphism* p *pulls back the Hamiltonian vector field of* H *to a positive multiple of* $X_{\tilde{H}_r}$,

$$p^* X_H = a X_{\tilde{H}_r} \text{ for } a > 0.$$

Furthermore, the flow of $X_{\tilde{H}_r}$ *is complete.*

In this theorem, the regularized Hamiltonian $\tilde{H}_r$ is actually the same for all energy levels. For positive mass ratio μ we have the following weaker statement.

Theorem B *After a proper time change, the flow on* $\Sigma^b_{\mu,c}$ *and on its double cover* $\tilde{\Sigma}^b_{\mu,c}$ *can be regularized, i.e. can be extended to a complete flow on a compact manifold, which we denote by* $\tilde{\Sigma}^{b,r}_{\mu,c}$. *Furthermore, for all* $c < L_1(0) = -3/2$ *and for all* $\epsilon(c) > 0$, *there is* $\mu_0(c) > 0$ *such that for all* $\mu \in [0, \mu_0(c)]$, *there is an embedding* $\tilde{\Sigma}^b_{\mu,c} \setminus U_{\epsilon(c)} \to \mathbb{R}^4$, *where* $U_{\epsilon(c)}$ *is an* $\epsilon(c)$*-neighborhood of the set of collisions, and a Hamiltonian* $\tilde{H}_r : \mathbb{R}^4 \to \mathbb{R}$ *with the following properties:*

(1) *the image* $i(\tilde{\Sigma}^b_{\mu,c})$ *is contained in a component of* $\tilde{H}_r^{-1}(c)$, *and the flow of* $X_{\tilde{H}_r}$ *is complete on this component.*
(2) *the Gauss-Kronecker curvature of* $i(\tilde{\Sigma}^b_{\mu,c} \setminus U_{\epsilon(c)})$ *is strictly positive.*
(3) *the Hamiltonian vector field* $X_{\tilde{H}_r}$ *is tangent to* $i(\tilde{\Sigma}^b_{\mu,c})$ *and is a reparametrization of* X_H *in the sense that the local diffeomorphism* p *pulls back the Hamiltonian vector field of* H *to a positive multiple of* $X_{\tilde{H}_r}$,

$$p^* X_H = a X_{\tilde{H}_r} \text{ for } a > 0.$$

In contrast to Theorem A, the regularized Hamiltonian of Theorem B depends on c; this phenomenon also happens when using Levi-Civita regularization and Moser regularization [26], [30].

Theorem A implies that the rotating Kepler problem is dynamically convex up to the first critical value, see Section 4 for the definitions. In particular, it reproves Theorem 1.1 in [3] avoiding direct index calculations. Furthermore, for T sufficiently large, Theorem A implies that for sufficiently small $\mu > 0$, the regularized bounded component is T-dynamically convex, *i.e.* all periodic orbits with period smaller than T have Conley-Zehnder index at least three. By the theory of holomorphic curves in symplectizations as established by Hofer, Wysocki and Zehnder [20], this implies the following corollary.

Corollary 1.1 *There is a continuous function $c(\mu)$ with $c(0) = -3/2$ such that for all $c < c(\mu)$, the bounded component of the regularized, restricted three-body problem around the heavy body, i.e. the flow of $\tilde{H}_r$ on $\tilde{\Sigma}^{b;r}_{\mu,c}$, admits a global disk-like surface of section after regularization.*

Details are given in Section 3.1. This particular corollary was already proved by McGehee with perturbative methods [29]. To motivate our alternative proof and indicate why one might care, we want to point out that our line of reasoning can work for other mass ratios as well; for example, see [2] or [28] for other work where convexity is applied to the restricted three-body problem. By contrast, McGehee's argument and also the argument in [3] relies on having an explicitly integrable flow, which one does not have for $\mu > 0$.

Seen from this point of view, Theorem A and Theorem B and the results in [2] make it seem reasonable to try to prove the general Birkhoff conjecture with holomorphic curves and convexity. This conjecture asserts that for all mass ratios $\mu \in [0, 1)$ and for all Jacobi energy below $L_1(\mu)$, the regularized restricted three-body problem admits a global disk-like surface of section in any bounded component of the energy hypersurface with double collisions regularized. The Birkhoff conjecture is already a hundred years old, and we briefly review some relevant background material in Section 4. The proof of Theorem A is given in Section 2. In Section 3 we extend this to the restricted three-body problem for small positive μ outside a neighborhood of the collisions, thus establishing Theorem B.

2. Proof of Theorem A: The rotating Kepler problem

In this section we will consider the Ligon-Schaaf regularization of the rotating Kepler problem, which we pull back with the Levi-Civita map. By computing the tangential Hessian, we will verify that this gives indeed a convex energy surface in $\mathbb{R}^4$, which proves Theorem A. We first review some special features of the rotating Kepler problem and the Ligon-Schaaf regularization.

2.1. A review of the Ligon-Schaaf regularization

The limit case $\mu = 0$ of the Jacobi Hamiltonian is known as the rotating Kepler problem, given by the Hamiltonian

$$H(q,p) = \frac{\|p\|^2}{2} - \frac{1}{\|q\|} + (q_1 p_2 - q_2 p_1).$$

This is the sum of the usual Kepler Hamiltonian in a fixed reference frame with an angular momentum term $(q_1p_2 - q_2p_1)$ generating the rotation of the reference frame. These two terms Poisson commute with each other, or put differently, this problem is rotationally invariant. Hence the rotating Kepler problem is completely integrable in the sense of Arnold-Liouville[1].

We note that the Hamiltonian of the rotating Kepler problem is singular due to two-body collisions: these correspond to orbits $t \mapsto (q(t), p(t))$ with $\lim_{t\to t_0} q(t) = 0$ for some t_0. We will now recall the Ligon-Schaaf regularization scheme as a way to regularize these collisions. Ligon and Schaaf discovered their regularization mapping [27] (anticipated by Fock [12]) in their attempt to understand the symmetries of the Kepler problem by the theory of moment maps. This regularization mapping can also be thought of as a global version of the Delaunay coordinate transformation. The somewhat mysterious properties of the Ligon-Schaaf regularization method still continue to fascinate mathematicians, see for example [10], [18].

Let us define this regularization. We will do this for the general n-dimensional Kepler problem, but shall only need the case $n = 2$ later. Denote the cotangent bundle of the n-sphere $\mathbb{S}^n$ by

$$T = T^*\mathbb{S}^n = \{(u,v) \in T^*\mathbb{R}^{n+1}; \|u\| = 1, u \cdot v = 0\}$$

and the deleted cotangent bundle of $\mathbb{S}^n$ by

$$T^\times = \{(u,v) \in T; v \neq 0\}.$$

The latter is sometimes called the Kepler manifold. Denote by P_- the subset

[1] In the slightly generalized sense that the gradients of the integrals are linearly independent almost everywhere rather than everywhere.

of $T^*\mathbb{R}^n$ with negative Kepler energy

$$H_0(p,q) = \frac{\|p\|^2}{2} - \frac{1}{\|q\|} < 0$$

and

$$T_- = \{(u,v) \in T^\times; u \neq (0,\ldots,0,1)\}.$$

To define the Ligon-Schaaf mapping, we put

$$\begin{aligned} \phi &= -\sqrt{-2H_0(q,p)}\langle q,p\rangle, \\ u &= \left(\sqrt{-2H_0(q,p)}\|q\|p, \|p\|^2\|q\| - 1\right), \\ v &= \left(-\|q\|^{-1}q + \langle q,p\rangle p, \phi\right). \end{aligned}$$

The vectors u and v are orthonormal vectors in $\mathbb{R}^{n+1}$, as can be checked with a direct computation. We treat the vector u as the base point in $\mathbb{S}^n$ and the vector v as a unit cotangent vector at u. The Ligon-Schaaf mapping is then given by

$$\begin{aligned} \Phi_{LS} : P_- &\longrightarrow T_- \\ (q,p) &\longmapsto \begin{pmatrix} r = (\cos\phi)u + (\sin\phi)v, \\ s = \dfrac{1}{\sqrt{-2H_0(q,p)}}(-(\sin\phi)u + (\cos\phi)v) \end{pmatrix}. \end{aligned}$$

It has been shown in [27], [10], [18] that this map is symplectic with respect to both canonical symplectic structures on cotangent bundles. Furthermore, it transforms $H_0(p,q)$ into the "Delaunay Hamiltonian"

$$H_k = -\frac{1}{2\|s\|^2}.$$

2.2. Application to the rotating Kepler problem

As we will only study the bounded component of the Hill's region (see Appendix A for a quick definition and overview) in which all motions are bounded, and thus with negative Keplerian energy, we may well restrict the rotating Kepler problem

$$H = \frac{\|p\|^2}{2} - \frac{1}{\|q\|} + (q_1 p_2 - q_2 p_1)$$

to P_-, as its dynamics is the composition of the dynamics of H_0 composed with a rotation. With the mapping Φ_{LS}, the Hamiltonian H is transformed into

$$H_r = -\frac{1}{2\|s\|^2} + (r_1 s_2 - r_2 s_1).$$

Both terms $-\frac{1}{2\|s\|^2}$ and $r_1 s_2 - r_2 s_1$ of H_r extend smoothly to the north pole $(0, \ldots, 0, 1)$ of $\mathbb{S}^n$ which represents the collisions, and the extensions are thus smoothly defined on $T^{\times}$.

On the other hand, in terms of the semi major axis a and the eccentricity e of the elliptic orbit, the Keplerian energy takes the value $H_0 = -\frac{1}{2a}$, with the norm of the angular momentum $|p_1 q_2 - p_2 q_1| = \sqrt{a}\sqrt{1-e^2}$. Moreover, as the bounded component of the Hill's region lies inside the circle $\{\|q\| = 1\}$, for all elliptic motions in this component we have $a < 1$. In conclusion, in the bounded component, we have

$$|r_1 s_2 - r_2 s_1| \leq \|s\| < 1.$$

From now on, we shall only consider the planar problem with $n = 2$. We have

$$T^*\mathbb{S}^2 = \{(r_1, r_2, r_3, s_1, s_2, s_3) \in \mathbb{R}^3 \times \mathbb{R}^3; \|r\| = 1, r \cdot s = 0\}$$

a point $(r_1, r_2, r_3, s_1, s_2, s_3)$ which is projected by stereographic projection to

$$(x_1, x_2, y_1, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{C} \times \mathbb{C} \ni (x = x_1 + ix_2, y = y_1 + iy_2)$$

such that

$$s_{1,2} = \left(\frac{\|x\|^2 + 1}{2}\right) y_{1,2} - Re(\bar{x}y) \cdot x_{1,2} \qquad s_3 = Re(\bar{x}y) \qquad r_{1,2} = \frac{2x_{1,2}}{\|x\|^2 + 1},$$

with the north pole projected to "the point at infinity" ∞. A computation shows that

$$\|s\|^2 = \frac{(\|x\|^2+1)^2}{4}\|y\|^2 \qquad r_1s_2 - r_2s_1 = x_1y_2 - x_2y_1.$$

Having in mind the switch in positions and momenta in the Moser regularization, see [30], which served as an intermediate step in Heckman-de Laat's interpretation [18] of the Ligon-Schaaf regularization, we take the following as our Levi-Civita mapping (conformally symplectic with a factor of 4)

$$\Phi_{LC} : T^*(\mathbb{C}\setminus 0) \to T^*\mathbb{C} \qquad (z,w) \mapsto (x = w/\bar{z}, y = 2z^2)$$

which can be smoothly extended to a mapping $T^*\mathbb{C}\setminus\{(0,0)\} \to T^*(\mathbb{C}\cup\{\infty\})$. The pull-back of H_r by Φ_{LC} thus reads

$$\tilde{H}_r := \Phi_{LC}^* H_r = -\frac{1}{2(\|w\|^2+\|z\|^2)^2} + 2(w_1z_2 - w_2z_1)$$

By the Ligon-Schaaf construction, see [27], [18] and [14, Chapter 4.3], and the fact that Levi-Civita regularization is a conformal symplectic map with constant factor 4, this Hamiltonian satisfies the claimed property (3) of Theorem A.

The corresponding energy level with energy c is therefore

$$\Gamma_c = \left\{-\frac{1}{2(\|w\|^2+\|z\|^2)^2} + 2(w_1z_2 - w_2z_1) = c\right\}.$$

Note that the angular momentum is $2(w_1z_2 - w_2z_1)$.

To verify property (1) of Theorem A, we make the following observations.

- By Proposition A.1, we know that $H^{-1}(c) \cap P_-$ has two components, $\Sigma_{0,c}^b$ and $\Sigma_{0,c}^u \cap P_-$
- The set $H_r^{-1}(c)$ has two components, namely one component corresponding to the regularization of $\Sigma_{0,c}^b$, and one component homeomorphic to $\Sigma_{0,c}^u \cap P_-$. We will call the regularized component $\Sigma_{0,c}^{b,r}$. As a topological space, $\Sigma_{0,c}^{b,r}$ is homeomorphic to $\mathbb{R}P^3$.
- The map Φ_{LC} is a $2-1$-covering map.

- The cover $(\tilde{\Sigma}^{b,r}_{0,c} := \Phi^{-1}_{LC}(\Sigma^{b,r}_{0,c}), \Phi_{LC}|_{\tilde{\Sigma}^{b,r}_{0,c}})$ is a connected cover. To see this, consider a point of the form $(z,0) \in \tilde{\Sigma}^{b,r}_{0,c}$. Then the path $t \mapsto (e^{\pi i t}z, 0)$ connects $(z,0)$ to $(-z,0)$, showing the claim.
- The same argument shows that $\Sigma^{u}_{0,c} \cap P_-$ has a connected cover, so Γ_c has two connected components[2], one compact (diffeomorphic to S^3) and one non-compact.

The fourth point shows that property (1) holds.

For property (2), we would like to understand if the bounded component $\Gamma_{0,c}$ (which corresponds to $\Sigma^b_{0,c}$) of Γ_c bounds a convex domain in $\mathbb{C}^2$. In order to show this, we calculate the Gauss-Kronecker curvature of $\Gamma_{0,c}$ and we show that this curvature is positive, which then implies that $\Gamma_{0,c}$ bounds a convex domain in $\mathbb{C}^2$ (see e.g. [14]).

For this purpose, it is enough to calculate the Hessian of the function

$$F := -1 + 4(w_1 z_2 - w_2 z_1)(\|w\|^2 + \|z\|^2)^2 - 2c(\|w\|^2 + \|z\|^2)^2.$$

restricted to the tangent space of $\Gamma_{0,c}$ and show that its determinant is positive. The set Γ_c is just the pre-image $F^{-1}(0)$ of 0.

To determine the normal direction of points on Γ_c, we calculate the gradient ∇F of F. We have

$$\nabla F = \big((\|w\|^2+\|z\|^2)g_1, (\|w\|^2+\|z\|^2)g_2, (\|w\|^2+\|z\|^2)g_3, (\|w\|^2+\|z\|^2)g_4\big)$$

with

$$\begin{aligned}
g_1 &= -4w_1^2 w_2 + 16 w_1 z_1 z_2 - 4w_2^3 - 20 w_2 z_1^2 - 4 w_2 z_2^2 - 8cz_1\\
g_2 &= 4w_1^3 + 4w_1 w_2^2 + 4w_1 z_1^2 + 20 w_1 z_2^2 - 16 w_2 z_1 z_2 - 8cz_2\\
g_3 &= 20 w_1^2 z_2 - 16 w_1 w_2 z_2 + 4w_2^2 z_2 + 4z_1^2 z_2 + 4z_2^3 - 8cw_1\\
g_4 &= -4w_1^2 z_1 + 16 w_1 w_2 z_2 - 20 w_2^2 z_1 - 4z_1^3 - 4z_1 z_2^2 - 8cw_2.
\end{aligned}$$

Note that we may naturally identify (g_1, g_2, g_3, g_4) with the quaternion $g := g_1 + g_2 i + g_3 j + g_4 k$. With this identification, we may thus find an

[2]For the restricted three-body problem one can, depending on the energy level, also get additional non-compact components around the other primary whose covers are not connected.

orthogonal frame of $T\Gamma_{0,c}$ by (right) multiplications with the quaternions i, j, k. Specifically, we may choose

$$v_1 = (-g_2, g_1, g_4, -g_3) \cong g \cdot i$$
$$v_2 = (-g_3, -g_4, g_1, g_2) \cong g \cdot j$$
$$v_3 = (-g_4, g_3, -g_2, g_1) \cong g \cdot k.$$

to form a basis of the tangent space at the point (w_1, w_2, z_1, z_2) provided the gradient

$$\nabla F = (\|w\|^2 + \|z\|^2)(g_1, g_2, g_3, g_4)$$

is non-vanishing. We now calculate the determinant DH of the restricted Hessian of F to the tangent spaces of $\Gamma_{0,c}$ and show it is positive. Note that this will also imply that ∇F is non-vanishing. Using the matrix representation $(v_1, v_2, v_3)^T \mathrm{Hess}(F)(v_1, v_2, v_3)$ for the tangential Hessian up to a factor of $\|g\|^6$, we directly compute the determinant

$$DH = \mathrm{Det}\big((v_1, v_2, v_3)^T \mathrm{Hess}(F)(v_1, v_2, v_3)\big).$$

This gives a somewhat unwieldy polynomial expression, which we will not write out here. Instead, we give the factorization of this multivariate polynomial, which can be obtained manually, as we shall do in Appendix B, or with a computer algebra program, such as Maple 18. The results we obtain either way agree of course. Indeed, in terms of $\bar{a} = \|w\|^2 + \|z\|^2$, $b = w_1 z_2 - z_1 w_2$ and c, we find the factorization

$$DH = 524288\bar{a}^6 f_1 f_2 f_3 f_4^2$$

where

$$f_1 = -2c + \bar{a} + 4b$$
$$f_2 = -2c - \bar{a} + 4b$$
$$f_3 = -4c^3 + 28bc^2 - (88b^2 - 7\bar{a}^2)c + 96b^3 - 15\bar{a}^2 b$$
$$f_4 = 4c^2 - 24bc + \bar{a}^2 + 32b^2.$$

Remark 2.1 Computer algebra programs include a multitude of factorization algorithms. Some simple versions of these algorithms are detailed in [15]: these algorithms have a completely mechanical nature. The derivation in the appendix relies more on ad hoc substitutions.

With the conditions $2|b| \le \bar{a} < 1$ and $-c > 3/2$, it is direct to see that

$$f_1 > 0, f_2 > 0, f_4 \ge 4(3b - c)^2 > 0.$$

We now show that under the same conditions, we also have $f_3 > 0$. For this, we substitute the relationship $b = \frac{1}{4\bar{a}^2} + c/2$ among $\bar{a}, b$ and c in the expression of f_3 and get

$$f_3 = \frac{12c^2\bar{a}^4 - 2c\bar{a}^8 - 15\bar{a}^6 + 14c\bar{a}^2 + 6}{4\bar{a}^6}.$$

In this expression the numerator is a quadratic function in c, whose graph is a parabola opening upward with as axis of symmetry the line $c = -\frac{7-\bar{a}^6}{12\bar{a}^2}$. For $\bar{a}^2 \ge 7/18$, we have

$$\frac{7 - \bar{a}^6}{12\bar{a}^2} < \frac{7}{12\bar{a}^2} < \frac{3}{2},$$

and hence this quadratic function is monotonically decreasing for $c < -3/2$. Its evaluation at $c = -3/2$ reads

$$3\bar{a}^8 - 15\bar{a}^6 + 27\bar{a}^4 - 21\bar{a}^2 + 6 = 3(\bar{a}^2 - 1)^3(\bar{a}^2 - 2)$$

which is clearly positive for $0 < \bar{a} < 1$. For $0 < \bar{a}^2 < 7/18$, we find that the evaluation of the numerator of f_3 at $c = -\frac{7-\bar{a}^6}{12\bar{a}^2}$ reads

$$-\frac{\bar{a}^{12}}{12} - \frac{83\bar{a}^6}{6} + \frac{23}{12},$$

which, as a quadratic equation in $\bar{a}^6$, is seen to be monotonically decreasing when $\bar{a}^6 > 0$. Moreover, its evaluation at $\bar{a}^2 = 7/18$ is seen to be positive (it is approximately 1.1028). This shows that f_3 is also a positive factor in the factorization of DH.

We have thus obtained the conclusion that $\Gamma_{0,c}$ bounds a convex domain for any energy c up to the first critical value $-3/2$. This proves Theorem A.

3. Proof and Discussions on Theorem B

To prove Theorem B, we will use that strict convexity is an open property, i.e. it is preserved under small perturbations. Unfortunately, the above construction involving the Ligon-Schaaf and Levi-Civita maps cannot be extended directly to perturbations of the rotating Kepler problem due to lack of smoothness of the perturbation term after Ligon-Schaaf regularization, and we are forced to leave out a neighborhood of the collisions.

Indeed, the Ligon-Schaaf regularization procedure does not involve a change of time parametrization, so we see directly that the Delaunay variables (L, l, G, g), defined by

$$\begin{cases} L = \sqrt{a} & \text{circular angular momentum} \\ l & \text{mean anomaly} \\ G = \pm L\sqrt{1-e^2} & \text{angular momentum, sign determined} \\ & \text{by the direction of motion} \\ g & \text{argument of pericentre,} \end{cases}$$

serve as a set of local coordinates in a neighborhood of collisions for the Ligon-Schaaf regularized (rotating) Kepler problem. A recent, detailed description of the Delaunay coordinates can be found in [1, Chapter 9.3] or in [32, 36]. The variables L, G, g depend only on the Keplerian orbit, and thus extend smoothly through collisions. The angle l now agrees with an angle parametrizing the great circle orbit of the geodesic flow on $T^{\times}$ and therefore extends smoothly. The neighborhood should be chosen not to contain circular motions on which the Delaunay variables are not all well-defined.

Now note that in the restricted planar circular three-body problem with mass ratio $0 < \mu \ll 1$, the Hamiltonian of the system can be decomposed as $K_{1-\mu} + P$, where the rotating Keplerian Hamiltonian $K_{1-\mu}$ with mass $1-\mu$ reads

$$K_{1-\mu} := \frac{\|p\|^2}{2} + (q_1 p_2 - q_2 p_1) - \frac{1-\mu}{\|q\|}$$

and the (singular) perturbation is given by

$$P = -\mu p_2 - \frac{\mu}{\|q - 1\|}.$$

A couple of observations are in order.

Lemma 3.1 *After the Ligon-Schaaf mapping and Levi-Civita regularization, the bounded component of the rotating Kepler Hamiltonian with mass* $1 - \mu$ *is convex below its critical value.*

Proof. Indeed, apply the linear transformation $f_{1-\mu} : (q, p) \mapsto \big((1-\mu)^{1/3} q, (1 - \mu)^{1/3} p\big)$. This map rescales the symplectic form and the Hamiltonian by a constant. In particular, level sets of $K_{1-\mu}$ are transformed to level sets of K_1, at a different energy level. The formula is $K_{1-\mu} \circ f_{1-\mu}(q, p) = (1 - \mu)^{2/3} K_1(q, p)$. As the level sets of the latter are convex after the procedure we described, so are the level sets of $K_{1-\mu}$. □

The second observation concerns the non-smoothness of the perturbation term at collisions near $q = 0$.

Once we switch to the Delaunay variables for the above-mentioned Kepler Hamiltonian, the dependence of the function P in the perturbing function on the mean anomaly l is not smooth at the collisions. This can be seen by the fact that the perturbing function is an analytic function of the Delaunay variables (L, G, g) and the eccentric anomaly u, via

$$q = e^{ig}\big(a(\cos u - e) + ia\sqrt{1 - e^2}\sin u\big),$$

with

$$a = L^2, \quad e = \sqrt{L^2 - G^2}/L.$$

We get from the Kepler equation

$$l = u - e \sin u$$

that

$$\frac{dl}{du} = 1 - e\cos u = \frac{\|q\|}{a},$$

which does not have a smooth inverse at the collisions $\|q\| = 0$.

To make an extension to the collision locus at $q = 0$ possible, it is necessary to change time from t to s according to the relation $dt/ds = \|q\|$. Making this Hamiltonian on a fixed energy surface is a standard procedure going back to Poincaré, [31] and a more recent description can be found in [30]. For completeness, we include this procedure here.

On a fixed energy surface $\{H - c = 0\}$, we multiply the shifted Hamiltonian $H - c$ by $\|q\|$ to obtain the new Hamiltonian $\|q\|(H - c)$. The associated Hamiltonian vector field gives rise to a smooth reparametrization of the flow on $\{H - c = 0\}$ outside $\|q\| = 0$; the reparametrization is singular on the collision locus at $q = 0$. We can now regularize the collisions by pulling back this Hamiltonian $\|q\|(K_{1-\mu} + P - c)$ under the Levi-Civita regularization mapping

$$\Phi_{LC} : T^*(\mathbb{C} \setminus 0) \to T^*\mathbb{C} \qquad (z, w) \mapsto (p = w/\bar{z}, q = 2z^2)$$

The resulting system is

$$\begin{aligned} K_{reg} &:= \Phi_{LC}^* \left(\frac{\|q\|}{2} (K_{1-\mu} + P - c) \right) \\ &= \frac{\|w\|^2}{2} - c\|z\|^2 + 2\|z\|^2(z_1 w_2 - z_2 w_1) + \|z\|^2 P(z^2) - \frac{1-\mu}{2}. \end{aligned}$$

Note that $\|z\|^2 P(z^2)$ is a smooth function of z near the regularized point $z = 0$, so this Hamiltonian is smooth on a neighborhood of $z = 0$. The non-smooth points of H correspond to $z = 1$ and any w. For the energy range that we consider, namely c less than the first critical value, we see that these points do not lie in the closure of the connected component of interest. We refer to [14, Chapter 5] for a detailed discussion of the topology of level sets of the restricted three-body problem. The upshot is that we may now extend the flow of the hypersurface $\{K_{reg} = 0\}$ through $\{z = 0\}$ to get a regular flow, meaning a flow generated by a non-vanishing vector field, on a closed manifold.

With this in mind, we conclude.

Proposition 3.1 *On its zero-energy surface, the Hamiltonian system defined by the Hamiltonian* $\Phi_{LC}^*(\|q\|(K_{1-\mu} + P - c))$ *extends to a regular system through collisions.*

Finally, as the Ligon-Schaaf regularization mapping Φ_{LS} is a symplectomorphism from P_- to T_-, and outside a neighborhood of the collisions, Φ_{LS} and Φ_{LS}^{-1} have bounded C^2 norms. Thus outside the image of this neighborhood, $\Phi_{LS}(\bar{\Sigma}_{\mu,c})$ is a smooth $O(\mu)$-deformation of $\Phi_{LS}(\bar{\Sigma}_{0,c})$. In particular, this image is therefore also convex for $\mu > 0$ small enough. This proves Theorem B.

3.1. Proof of Corollary 1.1

We first give a proof of the statement using known facts from the literature. Following [3] or more classically [31], there is a smooth family of periodic orbits $(\gamma_{0,c}, \tau_c)$ parametrized by c consisting entirely of non-degenerate orbits for $\mu = 0$, known as retrograde periodic orbits. For fixed c, the orbit τ_c has strictly minimal action[3] on the energy hypersurface $\tilde{\Sigma}^{b,r}_{0,c}$. Since non-degeneracy is an open condition, this family can be extended to small, positive μ: we write this two-parameter family as $(\gamma_{\mu,c}, \tau_{\mu,c})$. The property of minimal action of each $\gamma_{\mu,c}$ on each energy hypersurface $\tilde{\Sigma}^{b,r}_{\mu,c}$ also remains true, provided μ is sufficiently small and $c < L_1(\mu)$. Combining these observations we obtain μ_0, and $c(\mu)$ such each $\mu \in [0, \mu_0]$ and $c < c(\mu)$ lies in the Birkhoff set $\mathfrak{B}$ as defined in [14, Chapter 8.3.3]. To extend this function to larger μ, we observe that if c is very negative, then the Levi-Civita Hamiltonian is a convex function on a small ball as can be shown with the methods of [2]. Applying Theorem 4.3 and Theorem 4.1 gives us then a global disk-like surface of section for $\tilde{\Sigma}^{b,r}_{\mu,c}$ provided c is very negative. On the other hand, we can apply the proof of [14, Corollary 17.1.4] to obtain a global disk-like surface of section for $\tilde{\Sigma}^{b,r}_{\mu,c}$ as long as $c < c(\mu)$ and $\mu < \mu_0$. After extending the function $c(\mu)$ while combining the two statements, we obtain the claim of the corollary.

3.1.1 Sketch of an alternative argument using the convexity result

As in Theorem A we denote the the double cover of $\Sigma^b_{\mu,c} \subset H^{-1}(c)$ by $\tilde{\Sigma}^b_{\mu,c}$. After regularization, Theorem B gives us a Hamiltonian $\tilde{H}_r : \mathbb{R}^4 \to \mathbb{R}$. For $c < L_1(\mu)$, the level set $\tilde{H}_r^{-1}(c)$ has a compact component $\tilde{\Sigma}^{b,r}_{\mu,c}$, and the embedding i constructed in Theorem B sends $\tilde{\Sigma}^b_{\mu,c}$ into this compact component $\tilde{\Sigma}^{b,r}_{\mu,c}$.

[3]Recall that the action of a periodic Reeb orbit (γ, τ) of a contact form α is given by $\int_0^\tau \gamma^* \alpha$.

For $\mu = 0$ we can directly apply Theorem 4.1 and the methods involved to obtain a finite energy foliation of the symplectization of $\tilde{\Sigma}^{b,r}_{0,c}$. In particular, we obtain a binding orbit γ_0 with period T_0 (for a Reeb orbit the period equals the action). By the remarks following Theorem B, we know that all periodic Reeb orbits with period less than a sufficiently large period bound T (note that T can be increased arbitrarily by decreasing the parameter $\epsilon(c)$) have Conley-Zehnder index at least 3. We choose $T > 2T_0$, so by non-degeneracy of the so-called retrograde orbit (as classically proved in [31]; see [3] or [14] for more modern expositions), we know that this set of orbits is non-empty, i.e. there are periodic orbits with index at least 3 for sufficiently small $\mu > 0$. We now push this finite energy foliation obtained for $\mu = 0$ to positive μ. No breaking to orbits of index less than 3 can occur for action and energy reasons following the arguments of the proof of [14, Theorem 16.3.1]. We once again obtain a finite energy foliation, and the same arguments as in proof of [14, Corollary 17.1.4] imply the existence of a global disk-like surface of section for the regularized problem, i.e. for the flow of $\tilde{H}_r$.

Remark 3.1 This rather complicated argument has the advantage that it could potentially be applied to other setups.

4. Outlook: The Birkhoff conjecture

Birkhoff writes on page 328 of his seminal work on the restricted three-body problem [5]

> "This state of affairs seems to me to make it probable that the restricted problem of three bodies admit of reduction to the transformation of a discoid into itself as long as there is a closed oval of zero velocity about J(upiter) ..."

In modern mathematical language, "a transformation to a discoid" is referred to as the existence of a disk-like global surface of section. The assumption that there is a closed oval of zero velocity means that a bounded component of the restricted three-body problem for energies below the first critical value is considered. When Birkhoff referred to the restricted three-body problem, he assumed that it is regularized by Levi-Civita regulariza-

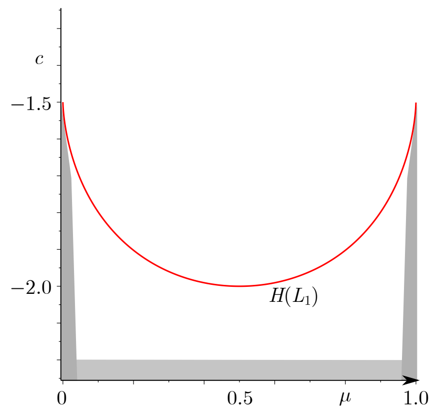

Figure 1. The gray-shaded regions indicate the parameters (μ, c), where the Birkhoff conjecture is known. Note that explicit values are not known, except for $\mu = 0$. We don't know whether there is any positive μ for which the Birkhoff conjecture is known up to the first critical value.

tion.[4] Therefore, when the energy is below the first critical value, the two bounded components are each diffeomorphic to a three dimensional sphere.

For small energies the Birkhoff conjecture is proved by Conley [9] and Kummer [23] for all mass ratios.

For sufficiently small mass ratios it was shown by McGehee in [29] that the Birkhoff conjecture holds true in the connected component around the heavy primary for an arbitrary energy below the first critical value. More precisely, he explicitly constructed a disk-like global surface of section for the rotating Kepler problem for Jacobi energies $c < -3/2$. For small $\mu > 0$, McGehee showed that there is a continuous function $c(\mu)$ with $c(0) = -3/2$ such that for all $c < c(\mu)$ there is a deformed disk-like global surface of section for the restricted three-body problem with mass ratio μ and Jacobi energy c. That similar results hold as well around the light primary was shown by Albers, Fish, Frauenfelder, Hofer, and van Koert in [2]. Figure 1 summarizes the parameters μ, c, where the Birkhoff conjecture is known to hold.

The proofs by Conley, Kummer and McGehee used perturbative meth-

[4]Levi-Civita published his paper on the regularization [26] in 1920, but had already announced his result in ICM 1904 [25]. Birkhoff referred to this regularization in his introduction. Goursat already anticipated this in the paper [17] published in 1887.

ods. By contrast, the proof in [2] is non-perturbative in nature. It uses global methods of modern symplectic geometry, namely the theory of holomorphic curves in symplectizations, due to Hofer [19] and Hofer-Wysocki-Zehnder [20]. Perturbative methods are only applicable if the system considered is close to a completely integrable system. This holds for small energy values, where the restricted three-body problem is a perturbation of the Kepler problem, and for small mass ratios around the heavy primary, where the restricted three-body problem is a perturbation of the rotating Kepler problem. However, for higher energies and higher mass ratios perturbative methods do typically not apply. We hope and expect that holomorphic curves will be the right way to attack the Birkhoff conjecture. For more information on the relation between the Birkhoff conjecture and the theory of holomorphic curves, we refer to [14].

In order to construct a disk-like global surface of section via holomorphic curves, the question about the existence of a convex embedding becomes crucial. The reasons are as follows. In [20], Hofer, Wysocki, and Zehnder proved the following result

Theorem 4.1 (Hofer-Wysocki-Zehnder) *Assume that Σ is a closed star-shaped hypersurface in $\mathbb{R}^4$ endowed with its standard symplectic structure. If Σ is dynamically convex, then Σ admits a disk-like global surface of section.*

Here a starshaped hypersurface in $\mathbb{R}^4$ is called dynamically convex if the Conley-Zehnder index of each closed characteristic is at least three. In [4], Albers, Frauenfelder, van Koert and Paternain proved that the Moser regularization of each bounded component of the restricted three-body problem for energies below the first critical value is fiberwise starshaped. This implies the following result.

Theorem 4.2 *The Levi-Civita embedding of each bounded component of the restricted three-body problem for energies below the first critical value is a starshaped hypersurface in $\mathbb{R}^4$.*

Proof. The argument that follows can also be found in [14, Chapter 4.2]. From [4] we know that the Liouville vector field $X = q \cdot \frac{\partial}{\partial q}$ is transverse to the energy hypersurface of RTBP. Recall that the Liouville vector field X solves the equation

$$\iota_X \omega_c = \lambda_c,$$

where ω_c and λ_c are the canonical symplectic form and canonical 1-form, respectively. Due to the switch map from Moser regularization, the roles of q and p are reversed, so these differential forms are given by

$$\omega_c = dq \wedge dp, \quad \lambda_c = q \cdot dp.$$

In complex coordinates we can write

$$\lambda_c = \Re(q d\bar{p}).$$

Pulling back λ_c under the Levi-Civita regularization with $y = q$ and $x = p$ gives us

$$\Phi_{LC}^* \lambda_c = \Re(2z d\bar{w} - 2\bar{w} dz).$$

This is 4 times the standard Liouville form on $\mathbb{C}^2$. We hence also have $\Phi_{LC}^* \omega_c = 4 \sum_j dz_j \wedge dw_j$, and we see that the pullback of X under the local diffeomorphism Φ_{LC} is the standard starshaped vector field

$$\sum_j z_j \frac{\partial}{\partial z_j} + w_j \frac{\partial}{\partial w_j}$$

It follows that the Levi-Civita embedding is a starshaped hypersurface in $\mathbb{C}^2 \cong \mathbb{R}^4$. □

In view of these results, in order to prove the Birkhoff conjecture, it suffices to show dynamical convexity for each bounded component of the restricted three-body problem for energies below the first critical value. However, to check dynamical convexity directly by first determining all closed characteristics, and then figuring out their Conley-Zehnder indices is in general not feasible. Instead of that, the following result of Hofer-Wysocki-Zehnder from [20] gives a much more handy approach (though other possibilities exist, e.g. [22]).

Theorem 4.3 (Hofer-Wysocki-Zehnder) *Assume that $\Sigma \subset \mathbb{R}^4$ is a closed strictly convex hypersurface. Then Σ is dynamically convex.*

This theorem explains the term "dynamical convexity". While dynamical convexity is a symplectic concept in the sense that it is preserved under symplectomorphisms, the notion of convexity is not. For a convex hypersurface in $\mathbb{R}^4$ there might well be a different starshaped embedding which is not convex.

Dynamical convexity of the rotating Kepler problem was proved already in [3] where it was also shown that the Levi-Civita embedding is not always convex for all energies. This result prompted the question if the rotating Kepler problem leads to an example of a dynamically convex hypersurface which does not admit a convex embedding at all. Theorem A answers this question in the negative. We mention here a recent result due to Chaidez and Edtmair, [6], which asserts that there are dynamically convex contact forms on S^3 that do not admit a convex embedding into $\mathbb{R}^4$.

We conclude this section with some other evidence for the Birkhoff conjecture.

(1) The Levi-Civita regularization of the restricted three-body problem is convex for a wide range of mass ratios and energies. This can be seen rather directly for very small energy. However, as was shown in [2] it also holds for energies close to the first critical value provided μ is close to 1.
(2) Numerically, we can verify the positivity of the tangential Hessian of the Levi-Civita regularization on a cover of the energy hypersurface. We then find that the Levi-Civita regularization seems to be convex for a wide range of energies. However, the Levi-Civita embedding does fail to be convex for energies very close to the critical energy when the mass ratio μ is close to 0.
(3) Again on the numerical side, it seems possible to adapt a shooting argument by Birkhoff into a numerical method to find periodic orbits carrying a certain reflection symmetry, to construct a parametrization of the retrograde orbit and a direct orbit. These orbits link as Hopf fibers, and numerical evidence suggests that it is possible to construct a global disk-like surface of section from these orbits.

The existence of a global surface of section reveals a lot about the orbit structure; it allows one to study the full dynamics with the globally defined return map, which can be shown to be conjugated to an area-preserving diffeomorphism. Since a lot is known about the dynamics of such maps, see for instance [13], [24], this provides ample means to better understand the dynamics.

Appendix A. Properties of the components of the unregularized and regularized problem

We consider the unregularized Jacobi Hamiltonian from Equation (1), which we rewrite as

$$H(q,p) = \frac{1}{2}\|p + Jq\|^2 - \frac{1-\mu}{\|q + (\mu, 0)\|} - \frac{\mu}{\|q - (1-\mu, 0)\|} - \frac{1}{2}\|q\|^2,$$

where we have defined

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We call

$$U(q) := -\frac{1-\mu}{\|q + (\mu, 0)\|} - \frac{\mu}{\|q - (1-\mu, 0)\|} - \frac{1}{2}\|q\|^2$$

the effective potential. We make the following observation: if $x = (q,p) \in H^{-1}(c)$, then we have

$$p = -Jq + \sqrt{2(c - U(q))} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

for some $\phi \in [0, 2\pi)$. We call the set $\{q \in \mathbb{R}^2 \mid c - U(q) \geq 0\}$ the Hill's region. By the above observation, we see that $H^{-1}(c)$ has as many components as the Hill's region. By [5, Section 7] or [14, Chapter 5.5], the Hill's region has three components for $c < L_1(\mu)$. We denote them as follows.

- we write $R^b_{-\mu}$ for the bounded component of the Hill's region whose closure contains the point $(-\mu, 0)$
- we write $R^b_{1-\mu}$ for the bounded component of the Hill's region whose closure contains the point $(1-\mu, 0)$.
- we write R^u for the unbounded component.

We write the corresponding components of $H^{-1}(c)$ as $\Sigma^{b,-\mu}_{\mu,c}$, $\Sigma^{b,1-\mu}_{\mu,c}$ and $\Sigma^u_{\mu,c}$. In the introduction we restricted ourselves to $\Sigma^{b,-\mu}_{\mu,c}$, which we more briefly denoted by $\Sigma^b_{\mu,c}$. We claim that all of these components are non-compact.

- for $\Sigma^u_{\mu,c}$, this follows from the observation that the q-coordinates on $\Sigma^u_{\mu,c}$: for example we may consider the sequence

$$\left\{ \left(\begin{pmatrix} n \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ n \end{pmatrix} + \sqrt{2(c - U(n,0))} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \right\}_{n=1}^{\infty}$$

  which is contained in $\Sigma^u_{\mu,c}$ and has unbounded q coordinates.
- for $\Sigma^{b,-\mu}_{\mu,c}$ and $\Sigma^{b,1-\mu}_{\mu,c}$ we can take the same formula, but we need to replace n by $-\mu + 1/n$ and $1 - \mu + 1/n$, respectively. This gives us sequences $\Sigma^{b,-\mu}_{\mu,c}$ and $\Sigma^{b,1-\mu}_{\mu,c}$ collapsing into the singularity at $(-\mu, 0)$ and $(1 - \mu, 0)$. In both cases the p coordinate diverges.

We verify non-convexity of each of these three components of $H^{-1}(c)$. For this, we note that for $c < L_1(\mu)$ and $\mu \in (0, 1)$, the equation

$$U(x, 0) = c$$

has six distinct solutions. We order these solutions as $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$.

- observe that $(x_1, 0)$ and $(x_6, 0)$ lie in the boundary of R^u. This gives us points $((x_i, 0); (0, x_i)) \in \partial\Sigma^u_{\mu,c}$ for $i = 1, 6$. However, the line segment is not contained in $H^{-1}((-\infty, c])$ nor is it contained in $H^{-1}([c, \infty))$. Indeed, the line segment is not contained in the domain of definition of H.
- the same argument can be applied to the points $(x_2, 0)$ and $(x_3, 0)$, which lie in the boundary of $\Sigma^{b,-\mu}_{\mu,c}$.
- similarly, this argument works for the points $(x_4, 0)$ and $(x_5, 0)$.

This shows non-convexity of all three components.

Remark A.1 The above argument constructs a line segment going through the singularity of H to show non-convexity. This is not the only way to see non-convexity, and alternatively, with more work we can compute the tangential Hessian to reach the same conclusion.

A.1. The rotating Kepler problem

We consider the limit case $\mu = 0$. In this case, the Jacobi Hamiltonian reduces to

$$H(q,p) = \frac{1}{2}\|p + Jq\|^2 - \frac{1}{\|q + (0,0)\|} - \frac{1}{2}\|q\|^2,$$

and the simpler effective potential

$$U_K(q) = -\frac{1}{\|q + (0,0)\|} - \frac{1}{2}\|q\|^2.$$

The same analysis as we did above for $\mu \in (0,1)$ yields the following. The equation

$$U_K(x,0) = c$$

has four distinct solutions for $c < L_1(0) = -3/2$, which we order as $x_1 < x_2 < 0 < x_3 < x_4$. We have $x_1 = -x_4$ and $x_2 = -x_3$.

- for $c < L_1(0) = -3/2$ the set $\{q \in \mathbb{R}^2 \mid c - U_K(q) \geq 0\}$ has two connected components. We write the bounded component as R^b. We have

$$R^b = \{q \in \mathbb{R}^2 \mid \|q\| \leq x_3, q \neq (0,0)\}.$$

  The unbounded component is

$$R^u = \{q \in \mathbb{R}^2 \mid \|q\| \geq x_4\}.$$

- similar to our earlier analysis, we have corresponding connected components of $H^{-1}(c)$. These are $\Sigma^b_{0,c}$ and $\Sigma^u_{0,c}$ corresponding to R^b and R^u, respectively.
- for non-convexity of any set bounded by $\Sigma^u_{0,c}$ we consider the points $((x_1,0);(0,x_1))$ and $((x_4,0);(0,x_4))$. The line segment connecting these points goes through the origin in $\mathbb{R}^4$, which, as before, is a singularity of H.
- for non-convexity of any set bounded by $\Sigma^b_{0,c}$ the same argument applies to the points $((x_2,0);(0,x_2))$ and $((x_3,0);(0,x_3))$.

We summarize the above and add a little additional information in the following proposition.

Proposition A.1 *For $c < L_1(0) = -3/2$, the rotating Kepler problem has* *two connected components, namely* $\Sigma^b_{0,c}$ *and* $\Sigma^u_{0,c}$*. This statement remains true if we restrict* H *to* P_-*. In fact, we have* $\Sigma^b_{0,c} \cap P_- = \Sigma^b_{0,c}$ *and* $\Sigma^u_{0,c} \cap P_-$ *is connected.*

Proof. We have shown the first assertion in the above discussion. For the second assertion, we note the following

- if $(q,p) \in \Sigma^b_{0,c}$, then $\|q\| < 1$, since $c < L_1(0) = -3/2$. It follows that $|L| < 1$ (for example by combining Formula 3.8 from [14] with the bound on $\|q\|$)). We see that the Kepler energy $\frac{1}{2}\|p\|^2 - 1/\|q\| < -1/2$ for points in $\Sigma^b_{0,c}$, so $\Sigma^b_{0,c} \cap P_- = \Sigma^b_{0,c}$.
- we show that $\Sigma^u_{0,c} \cap P_-$ is path-connected. Take a point x in $\Sigma^u_{0,c} \cap P_-$. We find a curve in $\Sigma^u_{0,c} \cap P_-$ connecting x to x'' in a standard position we define below. To do so, we first note from the above discussion that

$$x = \left(q, -Jq + \sqrt{2(c - U(q))} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \right)$$

for some $\phi \in [0, 2\pi)$. The angular momentum at x is

$$\begin{aligned} L(x) &= \left(q^t J + \sqrt{2(c - U(q))} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}^t \right) Jq \\ &= -\|q\|^2 + \sqrt{2(c - U(q))}(q_1 \sin\phi - q_2 \cos\phi). \end{aligned}$$

We will construct curves with constant Jacobi energy c along which angular momentum increases. This means that the Kepler energy decreases, so the curves lie in $\Sigma^u_{0,c} \cap P_-$. As a first step, we choose a curve starting in x with constant q, and for which ϕ changes monotonically, increasing L. The maximum angular momentum for constant q is achieved at

$$x' = \left(q, -Jq + \frac{\sqrt{2(c - U(q))}}{\|q\|} Jq \right).$$

By construction, the curve connecting x to x' lies in $\Sigma^u_{0,c} \cap P_-$. Now we choose a curve rotating q to the standard point $\binom{\|q\|}{0}$. The Kepler energy is fixed along this curve, so this curve lies in $\Sigma^u_{0,c} \cap P_-$, too. We end up with a point in standard position

$$x'' = \left(\begin{pmatrix} \|q\| \\ 0 \end{pmatrix} ; \left(\sqrt{2(c - U(\|q\|, 0))} - \|q\| \right) J \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

If we consider any other point y in $\Sigma^u_{0,c} \cap P_-$, we can bring this point into standard position, too,

$$y'' = \left(\begin{pmatrix} \|\tilde{q}\| \\ 0 \end{pmatrix} ; \left(\sqrt{2(c - U(\|\tilde{q}\|, 0))} - \|\tilde{q}\| \right) J \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).$$

Without loss of generality, we can and will assume that $\|\tilde{q}\| > \|q\|$. Connect x'' to y'' via a curve with increasing $\|q\|$. From the above form for the angular momentum, we see that the angular momentum is increasing along such a curve, so it lies in $\Sigma^u_{0,c} \cap P_-$. We conclude that $\Sigma^u_{0,c} \cap P_-$ is path-connected. □

Appendix B. Factorization of DH by hand

The factorization of DH can be done manually. The approach here was outlined by the original referee of an earlier version of the paper. Set

$$\alpha = \frac{1}{2}\|w\|^2 + \frac{1}{2}\|z\|^2, \quad \beta = \langle iw, z \rangle, \quad \gamma = c/2, \quad \omega = \beta - \gamma,$$

and

$$\delta = \frac{1}{2}\|w\|^2 - \frac{1}{2}\|z\|^2, \quad \epsilon = \langle w, z \rangle$$

so that

$$\epsilon^2 + \delta^2 = \alpha^2 - \beta^2.$$

We set $\mathbf{x} = (w, z) \cong w + zj \in \mathbb{C}^2 \cong \mathbb{H}$ and $\rho : \mathbb{C}^2 \cong \mathbb{H} \to \mathbb{C}^2 \cong \mathbb{H}$ is the involution $\rho(w, z) = (w, -z)$. We have

$$\langle \mathbf{x}, \rho(\mathbf{x}) \rangle = 2\delta, \quad \langle \mathbf{x}, \rho(\mathbf{x}) \cdot i \rangle = 0, \quad \langle \mathbf{x}, \rho(\mathbf{x}) \cdot j \rangle = 2\epsilon, \quad \langle \mathbf{x}, \rho(\mathbf{x}) \cdot k \rangle = 2\beta.$$

Therefore by rescaling by a factor of 1/16

$$F(\mathbf{x}) = -\frac{1}{16} + \alpha^2\omega, g(\mathbf{x}) = 2\omega\mathbf{x} + \alpha\rho(\mathbf{x}) \cdot k.$$

We thus have

$$\langle v, \rho(\mathbf{x}) \cdot k \rangle = -\frac{2\omega}{\alpha}\langle v, \mathbf{x}\rangle, \qquad v \in T_{\mathbf{x}}\Gamma_{0,c} \tag{*}$$

and

$$\langle v_1, \mathbf{x}\rangle = 2\alpha\epsilon, \qquad \langle v_2, \mathbf{x}\rangle = 0, \qquad \langle v_3, \mathbf{x}\rangle = -2\alpha\delta. \tag{**}$$

Moreover introduce

$$\phi_4 := \frac{1}{2\alpha}\|g\|^2 = \alpha^2 + 4(\omega^2 + \omega\beta),$$

which is just f_4 below, up to some constant.

Differentiating g in the direction $v \in \mathbb{C}^2$, one gets,

$$\nabla g \cdot v = 2\omega v + 2\langle \mathbf{x}, v\rangle \rho(\mathbf{x}) \cdot k + \langle \rho(\mathbf{x}) \cdot k, v\rangle \mathbf{x} + \alpha\rho(v) \cdot k.$$

Let $u := (-\epsilon, 0, \delta) \in \mathbb{R}^3$ and using (*), (**), we compute

$$U := \frac{1}{2\alpha}(v_1, v_2, v_3)^T \big(2\langle \mathbf{x}, \cdot\rangle \rho(\mathbf{x}) \cdot k + \langle \rho(\mathbf{x}) \cdot k, \cdot\rangle \cdot \mathbf{x}\big)(v_1, v_2, v_3) = -12\omega u \otimes u.$$

Define the matrix

$$D = \frac{1}{2\alpha}(v_1, v_2, v_3)^T \nabla g(v_1, v_2, v_3) = \frac{1}{2\alpha^2}(v_1, v_2, v_3)^T \nabla^2 F(v_1, v_2, v_3),$$

Where the second equality is established with the help of (*), which indicates that

$$\frac{1}{\alpha}\nabla^2 F - \nabla g = 2\omega\langle \mathbf{x}, \cdot\rangle \mathbf{x} + \alpha\langle \mathbf{x}, \cdot\rangle \rho(x) \cdot k = 0.$$

So D agrees with the Hessian of F up to a positive factor.

The matrix D can be decomposed as

$$D = 2\omega\phi_4 \mathrm{Id} + U + R,$$

where

$$R = \frac{1}{2}(v_1, v_2, v_3)^T (\rho(\cdot) \cdot k)(v_1, v_2, v_3) = \begin{pmatrix} -\phi_5 & -\phi_{12}\delta & 0 \\ -\phi_{12}\delta & \phi_5 & -\phi_{12}\epsilon \\ 0 & -\phi_{12}\epsilon & -\phi_5 \end{pmatrix}$$

in which

$$\phi_{12} = 4\omega^2 - \alpha^2, \qquad \phi_5 := \beta\phi_4 + 4\omega(\alpha^2 - \beta^2). \tag{***}$$

A computation shows that

$$\det D = d_1 + 12\omega\epsilon d_2 - 12\omega\delta d_3,$$

where d_1, d_2, d_3 are respectively the determinants of the matrices

$$D_1 = \begin{pmatrix} 2\omega\phi_4 - \phi_5 & -\phi_{12}\delta & 0 \\ -\phi_{12}\delta & 2\omega\phi_4 + \phi_5 & -\phi_{12}\epsilon \\ 0 & -\phi_{12}\epsilon & 2\omega\phi_4 - \phi_5 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} -\epsilon & 0 & \delta \\ -\phi_{12}\delta & 2\omega\phi_4 + \phi_5 & -\phi_{12}\epsilon \\ 0 & -\phi_{12}\epsilon & 2\omega\phi_4 - \phi_5 \end{pmatrix},$$

$$D_3 = \begin{pmatrix} 2\omega\phi_4 - \phi_5 & -\phi_{12}\delta & 0 \\ -\phi_{12}\delta & 2\omega\phi_4 + \phi_5 & -\phi_{12}\epsilon \\ -\epsilon & 0 & \delta \end{pmatrix}.$$

Using that $2\omega\phi_4 - \phi_5 = (\beta + 2\omega)\phi_{12}$ and computing these determinants using Laplace rule of expansion, we get

$$\det D = \phi_{12}^2(\beta + 2\omega)\Phi_4 - 12\phi_{12}\omega\epsilon^2\Phi_4 - 12\phi_{12}\omega\delta^2\Phi_4,$$

where

$$\Phi_4 := (\beta + 2\omega)(2\omega\phi_4 + \phi_5) - \phi_{12}(\alpha^2 - \beta^2).$$

Using both identities in (***), we find $\Phi_4 = \phi_4^2$. Thus we conclude

$$\det D = \phi_{12}\phi_3\phi_4^2$$

where $\phi_3 = \phi_{12}(\beta + 2\omega) - 12\omega(\alpha^2 - \beta^2)$ which is equal to f_3 up to a constant factor.

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